On the Response of Linear Systems to Signals Modulated in Amplitude and Frequency*

Zweig, Schultheiss, and Wogrin[1] have obtained an expression for the response of a passive network, specified by its complex admittance, to a wave simultaneously frequency and amplitude modulated. For this purpose they make use of the somewhat elaborate analysis due to Carson and Fry.[2] It is of interest to notice that their final expression can be written down almost by inspection, following a method developed by Bloch.[3]

Expand the network admittance, $Y(i\omega)$, as a power series in the departure, ν, from ω, so

$$Y[i(\omega + \nu)] = \sum_{n=0}^{\infty} \frac{(i\nu)^n}{\underline{|n}} \frac{1}{(i)^n} \frac{d^n}{d\omega^n} Y(i\omega). \quad (1)$$

Now, it must be possible to express the applied voltage, $F(t)$, as a set of tones (sidebands), so that

$$F(t) = \sum_{\nu} f(\nu) \exp[i\omega t + i\nu t + \phi_\nu] = \exp(i\omega t) \sum_{\nu} f(\nu) \exp[i\nu t + \phi_\nu]. \quad (2)$$

Each tone in (2) can now be multiplied separately by the appropriate value of $Y[i(\omega + \nu)]$, so that a representative tone becomes from (1),

$$\exp(i\omega t) \sum_{n=0}^{\infty} \frac{1}{\underline{|n}(i)^n} (i\nu)^n f(\nu) \cdot \exp[i\nu t + \phi_\nu] \frac{d^n}{d\omega^n} Y(i\omega)$$
$$= \exp(i\omega t) \sum_{n=0}^{\infty} \frac{1}{\underline{|n}(i)^n} \left\{ \frac{d^n}{dt^n} [f(\nu) \cdot \exp(i\nu t + \phi_\nu)] \right\} \frac{d^n}{d\omega^n} Y(i\omega).$$

Thus, using (2), the output current $I(t)$ is given by

$$I(t) = \exp(i\omega t) \left\{ \sum_{n=0}^{\infty} \frac{1}{\underline{|n}} \frac{1}{(i)^n} \frac{d^n}{dt^n} \cdot [\exp(-i\omega t) F(t)] \frac{d^n}{d\omega^n} Y(i\omega) \right\}. \quad (3)$$

If one introduces the functions $C_n(t)$ by writing

$$\exp\left[i \int_{-\infty}^{t} \mu(x)\, dx \right] C_n(t) = \frac{1}{(i)^n} \frac{d^n}{dt^n} [\exp(-i\omega t) F(t)]$$

it is readily seen that (3) is equivalent to (17) of Zweig, *et. al.*[1] The forms of the expressions $C_n(t)$, given as (18) to (21) in Zweig, *et. al.*[1], are now obtainable by carrying out the indicated differentiations.

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[1] F. Zweig, P. M. Schultheiss, and C. A. Wogrin, "On the response of linear systems to signals modulated in both amplitude and frequency," IRE Trans., vol. CT-2, pp. 367–369; December, 1955.

[2] J. R. Carson and T. C. Fry, "Variable-frequency electric circuit theory with application to the theory of frequency modulation," *Bell Sys. Tech. J.*, vol. 16, pp. 513–540; October, 1937.

[3] A. Bloch, "Modulation theory," *J. I.E.E.* vol. 91, pt. III, pp. 31–42; March, 1944.

On the Design of Active Filters with Butterworth Characteristics*

The need frequently arises for a low-pass filter circuit using no inductors, and suitable for inclusion in a direct-coupled amplifier chain. Expediency usually dictates the use of the circuit of Fig. 1. It appears, however,

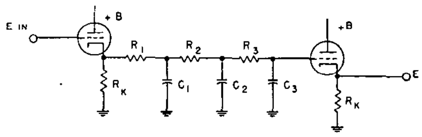

Fig. 1—Conventional low-pass filter.

that a feedback arrangement using the same number of elements is capable of better performance. In particular, a circuit using only resistors, capacitors, and cathode followers can provide the second and third-order Butterworth approximation to the ideal low-pass filter characteristic.[1]

The circuit of Fig. 2 has the flow graph[2] of Fig. 3. The cathode-followers are assumed to have unity gain and zero output impedance. Reduction of the flow graph leads to the expression for the gain of the entire circuit

$$\frac{E_0}{E_{in}} = \frac{1}{1 + R_2C_2s + R_1C_1R_2C_2s^2}.$$

If R_2C_2 is chosen to be 1.414 seconds, and R_1C_1 is made 0.707 seconds, the denominator of the above expression becomes $s^2 + 1.414s + 1$, making the circuit a Butterworth filter of order two. If the circuit of Fig. 4 is used with RC chosen as 1.0 second, the denominator of the gain expression becomes $(s^2 + s + 1)(s + 1)$, making the circuit a Butterworth filter of order three.

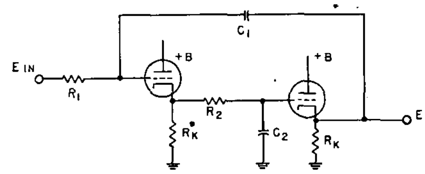

Fig. 2—Feedback low-pass filter.

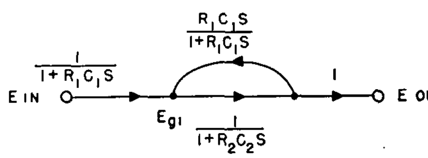

Fig. 3—Signal flow graph.

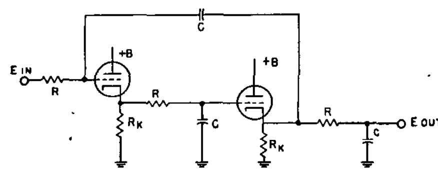

Fig. 4—Third-order Butterworth filter.

In practice these filters are very attractive. They are as stable as the cathode followers themselves. They can be included easily in a direct-coupled amplifier. They can be scaled to any desired cutoff frequency simply by dividing the time constants given above by the cutoff frequency (in radians per second) desired, and for the special case of the Butterworth low-pass filter, their design is simpler than that of Sallen and Key.[3]

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[1] R. F. Baum, "A contribution to the approximation problem," Proc. IRE, vol. 36, pp. 863–869; July, 1948.

[2] S. J. Mason, "Feedback theory—some properties of signal flow graphs," Proc. IRE, vol. 41, pp. 1144–1156; September, 1953.

[3] R. P. Sallen and E. L. Key, "A practical method of designing rc active filters," IRE Trans., vol. CT-2, pp. 74–85; March, 1955.