at 9,090 megacycles to be given for susceptance between 0 and -6 by

$$B = -\frac{\lambda_g}{a}\cot^2\frac{\pi d}{2a}\cdot$$

In this formula d is the distance between center lines of the posts decreased by 0.094 inch, and a is the inner width of the guide.

The center line spacing of a pair of posts to give a susceptance of -2.0 was found. Then a single stage was constructed of two pairs spaced three-eighths of wavelength apart, and its Q was measured as 8.5. (See Table I.) It was necessary to lengthen the spacing between pairs by about 0.010 inch to correct for the thickness of the posts.

In the computation of the spacing for critical coupling equation, only the lowest mode is assumed to be present. From (5) the spacing between stages should be either one-eighth or five-eighths of a guide wavelength, and two-stage filters were made to these specifications. When the spacing was the lesser of these values (0.233 inch), the plot of standing-wave ratio as a function of frequency showed a match on either side of resonance, but the standing-wave ratio at resonance was 2.3 decibels (in voltage 1.32). This high standing wave is presumably due to higher mode interaction, since when the spacing between stages was lengthened by a half wavelength; the standing-wave ratio at resonance dropped to 0.1 decibel (in voltage 1.02), and the curve was flat at this point. The measurement of Q's was mentioned earlier.

Continuously Adjustable Electronic Filter Networks*

GLENN E. TISDALE†, ASSOCIATE MEMBER, IRE

Summary—**Standard network theory is used to explain the advantages of *RC* networks in the design of filters with frequency response characteristics which are continuously adjustable over several decades. The theoretical discussion is followed by a review of some simple circuits from which most filter characteristics may be obtained. The shape of the response as a function of frequency is determined largely by passive networks. Tubes serve to isolate the passive networks and, in some cases, to invert their response by means of feedback amplification. An example of the method is demonstrated by the design of an adjustable low-pass high-pass filter.**

Review of Theory

THE DIMENSIONLESS complex ratio of the output voltage to the input voltage of any four-terminal network can be written[1] in terms of the complex frequency, $p=\sigma+j\omega$, as

$$F(p) = \frac{a_m p^m + a_{m-1}p^{m-1} + \cdots + a_1 p + a_0}{b_n p^n + b_{n-1}p^{n-1} + \cdots + b_1 p + b_0}\cdot \quad (1)$$

This quotient of two polynomials in p can be called the transfer function of the network. The constants a and b are real in physical circuits, but are not necessarily positive. If (1) is factored to give zeros p' and poles p'', the result has the form

$$F(p) = \frac{a_m(p - p_1')(p - p_2')\cdots(p - p_m')}{b_n(p - p_1'')(p - p_2'')\cdots(p - p_n'')}\cdot \quad (2)$$

Because the coefficients of p in (1) are real, any complex or imaginary zeros or poles of $F(p)$ must occur in conjugate pairs. In addition, the poles must have negative, nonzero real parts,[2] i.e., they fall in the left half of the p plane. Zeros may fall anywhere in the p plane.

Except for the output level, a_m/b_n in (1), any stable characteristic can be obtained with passive networks. This is stated as a theorem by Bode,[3] and it eliminates speculation about new filter characteristics possible with the use of tubes. Tube parameters, such as the amplification factor, cannot usually be specified closely, nor can they be varied widely without the appearance of nonlinear phenomena. The shape of the response characteristics as a function of frequency will therefore be derived from passive sections. Tubes will serve only to isolate sections, to raise the output level, or, as will be shown later, to invert certain transfer functions by means of feedback amplification.

The necessary conditions for a shift of the attenuation and phase characteristics of a transfer function intact along the logarithmic frequency scale can be obtained from (1) and (2). A response in the sum form of (1) is shifted by a factor s if each of its terms is multiplied by s raised to the power of p in that particular term. If s is greater than unity, the response is shifted to a lower frequency. The same shift in frequency is obtained by a division by s of the poles and zeros of the factored expression of (2). A dimensional analysis of $F(p)$ will now be used to realize this frequency shift by a variation of the circuit parameters of physical networks. It will be shown that RC networks are particularly suitable for this purpose at low frequencies.

Equation (1) may be factored to give

$$F(p) = \left(\frac{a_0}{b_0}\right)\frac{\alpha_m p^m + \alpha_{m-1}p^{m-1} + \cdots + \alpha_1 p + 1}{\beta_n p^n + \beta_{n-1}p^{n-1} + \cdots + \beta_1 p + 1}, \quad (3)$$

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† Formerly, Yale University, New Haven, Conn.; now, Servo Corporation of America, New Hyde Park, L. I., N. Y.

[1] H. W. Bode, "Network Analysis and Feedback Amplifier Design," D. Van Nostrand Co., New York, p. 25; 1945.

[2] See p. 111 of footnote reference 1.

[3] See p. 245 of footnote reference 1.

where $\alpha_k = a_k/a_0$ and $\beta_k = b_k/b_0$ can be considered as functions of passive elements with no loss in generality, as stated previously. It is seen that a_0/b_0 is dimensionless, since $F(p)$ is the ratio of two voltages. Likewise, all the terms of the quotient are dimensionless, since they form a series with unity. As p has the dimension of the reciprocal of time, or $1/[T]$, the dimensions of α_k and β_k are $[T]^k$.

By definition, the dimensions of the three circuit elements are

$$\begin{aligned}
&\text{Resistance, } R \quad \frac{\text{EMF}}{\text{Current}} \quad \text{or} \quad [R];\\
&\text{Capacitance, } C \quad \frac{\text{Current-Time}}{\text{EMF}} \quad \text{or} \quad \frac{[T]}{[R]};\\
&\text{Inductance, } L \quad \frac{\text{EMF-Time}}{\text{Current}} \quad \text{or} \quad [R][T].
\end{aligned} \tag{4}$$

For the purposes of this discussion the symbols $[R]$, $[L]$, and $[C]$ are defined as any mathematical combination of the corresponding circuit elements having the dimensions of a single element of that kind. Thus the expressions $C_1+C_2+C_3$ and C_1C_2/C_1+C_2 are both represented by $[C]$. In (3) for $F(p)$ the dimensions $[T]^k$ for α_k and β_k can be obtained by using relations (4) from the formulas

$$\left.\begin{matrix}[\alpha_k]\\ [\beta_k]\end{matrix}\right\} = [C]^{k-j}[L]^j[R]^{k-2j} = [T]^k. \tag{5}$$

Here j is an integer of either sign.

It was stated previously that the characteristic $F(p)$ may be shifted along the frequency scale by a factor s if each coefficient α_k and β_k of p^k is divided by s^k. This division can be associated with the terms of (5) as follows:

$$\frac{[C]^{k-j}}{s^{k-j}} \cdot \frac{[L]^j}{s^j} \cdot [R]^{k-2j}. \tag{6}$$

This result indicates that a sufficient condition for the frequency shift of a characteristic is the alteration of the values of all capacitances and inductances in the circuit by the reciprocal of the shifting factor. Resistances, being frequency insensitive, remain fixed. Unfortunately, this result is often impractical at low frequencies. An attempt to vary R and C instead of L and C is, in general, unsuccessful.

If no inductances are present, $j=0$, and

$$\left.\begin{matrix}[\alpha_k]\\ [\beta_k]\end{matrix}\right\} = [C]^k[R]^k. \tag{7}$$

If no capacitances are present, $j=k$, and

$$\left.\begin{matrix}[\alpha_k]\\ [\beta_k]\end{matrix}\right\} = [L]^k[R]^{-k}. \tag{8}$$

In either of these expressions the frequency shift can be made by a variation of all values of any single parameter. At low frequencies the RC function with variable resistors is usually most suitable. Because all resistors must be variable, circuit design should include a minimum of this type of element. It is therefore of importance to know the relation between the number of resistors and the degree of the numerator and denominator of $F(p)$; in other words, the number of available zeros and poles.

It can be shown by the methods of matrix algebra that the largest number of poles and zeros of $F(p)$ is less than or equal to the number of separate resistors in an RC network. This statement is also true for the number of separate capacitors. In general, therefore, RC networks having an equal number of resistors and capacitors will present a given number of poles and zeros most economically. An effect of extra capacitors is discussed later.

In establishing a method for the design of a general filter characteristic using RC networks, a difficulty arises due to the restriction on the location of the poles. While zeros may appear anywhere on the complex plane, the poles are always negative-real.[4] This restriction is removed by the introduction of vacuum tubes.

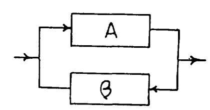

Fig. 1—Representation of feedback amplifier.

Complex poles are obtained if a network having equivalent complex zeros is used as the β circuit of a simple feedback loop. For the feedback amplifier of Fig. 1 it can be shown[5] that

$$F(p) = \frac{A}{1 + A\beta} \approx \frac{1}{\beta}, \tag{9}$$

if the gain A of the amplifier is sufficiently large that $A\beta \gg 1$. Then any zeros of the β circuit will become poles, and vice versa, within the validity of the approximation.

Design Procedure

The design of an electronic RC filter network can begin with the statement of the frequency characteristic in the form of a transfer function, factored to give poles and zeros. The function is then broken up into a number of separately realizable factors. The networks from which the factors are realized are joined by cathode followers or other isolating devices to give the original function.

A variety of more or less familiar circuits can be used to obtain particular configurations of poles and zeros. Simple ladder structures provide poles and zeros on the real axis and zeros at the origin. Zeros on the imaginary axis can be obtained from the parallel T and similar

[4] E. A. Guillemin, "Communications Networks," John Wiley and Sons, Inc., New York, N. Y., vol. II, p. 208; 1935.

[5] MIT Staff, "Applied Electronics," John Wiley and Sons, Inc., New York, N. Y., p. 526; 1943.

structures.[6] These same structures yield complex zeros, but as their transfer functions are cubic in form, the design problems are considerable. Complex zeros in the left-half plane are more readily obtained from the bridged T of Fig. 2 for which the polynomials in $F(p)$ are quadratic. If the transfer function for this network

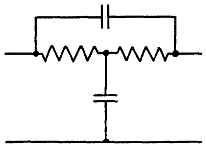

Fig. 2—Bridged-T network.

is inverted in a feedback loop, the complex zeros become a pair of complex poles. A capacitor shunted across the output of the bridged T shifts the poles without affecting the zeros. This example of the effect of extra elements of one kind in an RC network also provides information on the changes in the characteristic due to the loading of each network by the grid-to-ground capacities of the isolating stages.

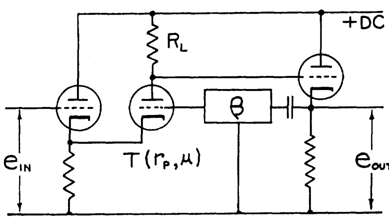

Fig. 3—Practical form of feedback amplifier for μ-circuit inversion.

A practical circuit[7] for the inversion of a β circuit is shown by Fig. 3. The transfer function of this circuit is

$$F(p) = \frac{1+\mu}{1+\dfrac{r_p + R_k(1+\mu)}{R_L} + \mu\beta} \approx \frac{1}{\beta}, \tag{10}$$

if the amplification factor μ of amplifier T is many times greater than unity, and if the effective cathode impedance R_k presented to T is many times less than R_L. From the exact expression it is seen that the number of zeros cannot exceed the number of poles, regardless of the form of β.

Adjustable Low-Pass High-Pass Filter

An example of the method is the development of an adjustable electronic filter having the frequency characteristics of the prototype low-pass section of Fig. 4. This network has a flat response in the pass band and an attenuation rate of 18 db per octave well above the cutoff frequency. The transfer function is

$$F(p) = KF_1(p)F_2(p)$$

$$= \frac{K}{\left(1+\dfrac{p}{\omega_k}\right)\left(1+2\zeta\dfrac{p}{\omega_n}+\left(\dfrac{p}{\omega_n}\right)^2\right)} \tag{11}$$

The constants K, ω_k, ω_n, and ζ are real and positive, and $\zeta<1$, which implies a pair of complex poles. Reasonable design values for a flat pass band and sharp cutoff are $\zeta=0.35$ and $\omega_k/\omega_n=0.707$. Reduction of ζ sharpens the cutoff at the expense of flat pass response.

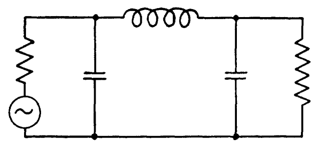

Fig. 4—Prototype low-pass section.

The real pole of $F_1(p)$ is obtained by a single RC section. Inversion of a bridged T yields the complex poles of $F_2(p)$, and in addition a pair of real zeros, which must be cancelled by a matching pair of real poles. The complete circuit is shown by Fig. 5. In the electronic network $K=1$, which implies essentially unity transmission for this section below cutoff. In the prototype section, $K<1$.

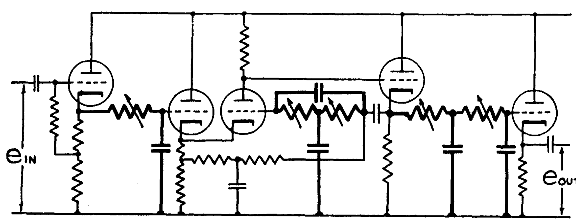

Fig. 5—Continuously adjustable low-pass filter. The RC networks which determine the frequency characteristics are drawn with heavy lines. The variable resistors are ordinarily gauged.

The high-pass equivalent of Fig. 4 is obtained by the substitution of ω_n/p for p/ω_n in (11). The result is

$$G(p) = \frac{K\dfrac{p^3}{\omega_n{}^3}}{\left(\dfrac{p}{\omega_n}+\dfrac{\omega_n}{\omega_k}\right)\left(1+2\zeta\dfrac{p}{\omega_n}+\left(\dfrac{p}{\omega_n}\right)^2\right)}. \tag{12}$$

If $\omega_k=\omega_n$, $G(p)$ is obtained from the low-pass expression by the addition of three zeros at the origin. If $(\omega_k/\omega_n)\neq 1$, the shape of the high-pass and low-pass curves will be mirror images about ω_n if the two values for this ratio are reciprocal. The zeros are introduced by reversing the positions of R and C in the ladder sections of Fig. 5. The inverted β circuit is unchanged. A few switches permit the circuit to perform as a high-pass or a low-pass filter.

Acknowledgment

The author is indebted to W. J. Cunningham for helpful suggestions made during the course of this investigation.

[6] G. R. Harris, "Bridged reactance-resistance networks," Proc. I.R.E., vol. 37, pp. 882–887; August, 1949.
[7] G. E. Valley and H. Wallman, "Vacuum Tube Amplifiers," McGraw-Hill Book Co., New York, N. Y., p. 402; 1948.