An iterative analogue computer for use with resistance network analogues.

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When field problems involving sources or sinks are solved using resistance network analogues, currents must be applied to the nodes of the network such that each current is a desired function of the associated node potential.

This requires repeated re-adjustment of every current until each is correct. A method is described whereby the adjustments are made automatically by a scanning switch, a function generator, an a.c. amplifier, and a set of triode valves each connected to a network node and having a memory capacitor connected to its grid. The solution is thus achieved rapidly, and changing problems can be solved continuously.

Suitably modified, the apparatus may find uses such as solving other differential equations or providing a set of non-linear resistors for fluid flow analogues.

Partial differential equations of the type

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = f(\phi) \qquad (1)$$

describe the distribution of a variable ϕ in a two-dimensional field from which a quantity $f(\phi)$ is withdrawn per unit area. In principle they can be solved when the boundary conditions are given, but in practice an algebraic solution is often difficult or impossible.

As Liebmann[1,2] has shown, they can be solved with quite high accuracy by the use of a network of resistors to simulate the medium in which the field exists. Potentials corresponding to the boundary values of ϕ are applied to the boundaries of the network, and currents corresponding to $f(\phi)$ are withdrawn from the nodes. The resulting distribution of potential over the network gives the distribution of ϕ over the field. The network may be one-, two- or three-dimensional, but most problems can be reduced to one or two dimensions and consideration will be limited to these cases.

Adjustment of any one current modifies the potentials at all the nodes, and hence the (interim) desired values of all the other currents. It is therefore necessary to repeat all the adjustments several times until the desired relation between current and potential holds at every node.

This procedure has been made easier in the case where $f(\phi) = - K^2\phi$ by a method[3] in which the errors in adjustment are displayed on an oscilloscope by means of a scanning switch. It is then possible to see which are the worst errors and to reduce them first and by the (interim) optimum amount.

In another method, described by Karplus,[4] the current adjustments are made automatically by conventional analogue computer units connected to the nodes of the network and maintaining continuously the desired relationship between the currents and the potentials. Each computer unit generates the desired function independently, and there are as many units as there are nodes: the method therefore becomes somewhat expensive if extended to two-dimensional networks.

Automatic methods of solution are valuable, however, since they may considerably extend the scope of resistance network analogues by permitting the continuous solution of changing problems, or the determination of critical values (eigenvalues) of parameters which give particularly significant solutions.

The method to be described provides for the currents to be adjusted automatically one after another by means of a single function generator, a scanning switch and an a.c. amplifier. These work in conjunction with a set of triode valves, each connected to a node of the network and controlling the current applied to it. Each triode possesses a "memory" capacitor connected to its grid whose potential is adjusted once every cycle in relation to the node potential and remains steady in between adjustments. Any desired relationship can be obtained between current and node potential by adjusting the function generator, and this item may economically be made accurate since only one is used. The only moving part is the switch, and if this is rotated at suitable speed, the solution can be reached in a few seconds or less. The amplifier is simple, and the triode characteristics are unimportant. The method has been used experimentally with one-dimensional networks and should extend readily to two dimensions.

PRINCIPLE OF THE ANALOGUE

Consider for simplicity the one-dimensional case. Equation (1) may be written in finite difference form:

$$\frac{d^2\phi}{dx^2} = \frac{\phi_{n+1} - \phi_n}{h^2} - \frac{\phi_n - \phi_{n-1}}{h^2} = f(\phi_n) \qquad (2)$$

where h is the interval in x (the mesh interval) between adjacent nodes. In this case the analogue is a line of equal

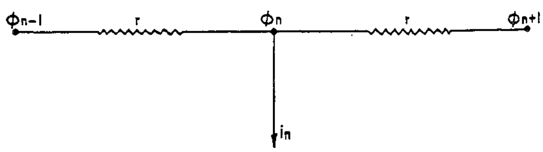

Fig. 1. Typical node of a one-dimensional network

resistors r, represented by a typical node in Fig. 1. If current i_n is withdrawn from a node at potential ϕ_n, then

$$\frac{\phi_{n+1} - \phi_n}{r} - \frac{\phi_n - \phi_{n-1}}{r} = i_n \qquad (3)$$

which is exactly analogous to equation (2) if

$$i_n = \frac{h^2}{r} f(\phi_n) \qquad (4)$$

Equation (2) can therefore be solved by the circuit of Fig. 2 if the grid potential of each triode is adjusted so that the potential V_n across its cathode resistor R is

$$V_n = Ri_n = h^2\frac{R}{r}f(\phi_n) \quad (n = 1, 2 \ldots 5) \tag{5}$$

ϕ_0 and ϕ_6 are applied boundary potentials, V_b is a common h.t. supply, and the solution is given by the values of ϕ_{1-5} along the network. h in this case is 1/6th the range of x considered, but closer spacing would be used in most practical problems. Boundary currents may be specified instead of potentials.

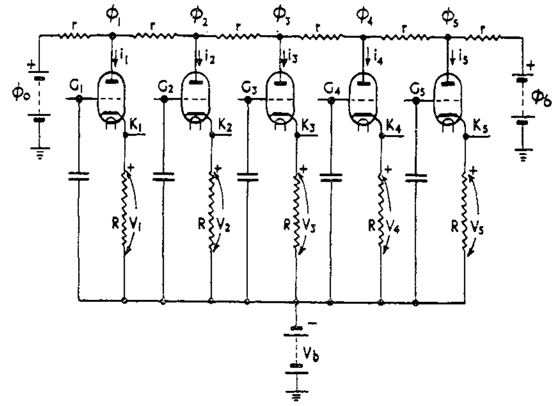

Fig. 2. Network with triodes to control (positive) currents

The circuit of Fig. 2 deals only with cases where i and hence $f(\phi)$ are always positive. If they are always negative the same circuit suffices with the polarities of all ϕ reversed and

$$V_n = -\, h^2\frac{R}{r}f(\phi_n) \tag{6}$$

If $f(\phi)$ changes sign, so must i and reversible current-controlling elements are required as typified in Fig. 3.

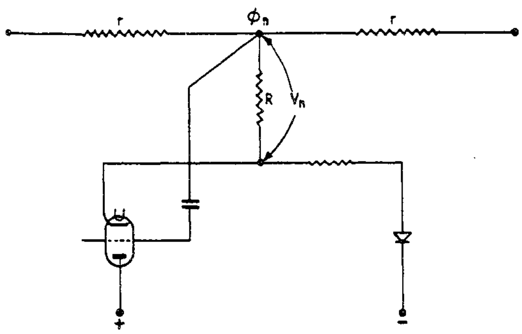

Fig. 3. Reversible control element

METHOD OF AUTOMATIC CURRENT ADJUSTMENT

Fig. 4 shows a system which automatically adjusts and re-adjusts the grid potentials of the triodes until the current through each is correct. It applies to the circuit of Fig. 2 where $f(\phi)$ is always positive, a modified arrangement being required for adjusting reversible elements such as shown in Fig. 3.

The section S_ϕ of a ganged three-pole switch presents the node potentials in turn to a function generator F as the switch is continuously rotated. The outputs $F(\phi)$ are applied to one input terminal of an amplifier A whose other input

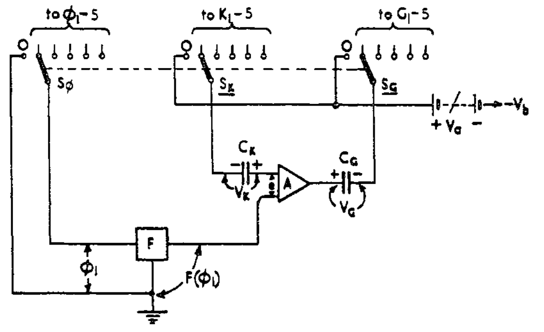

Fig. 4. Scanning and current-adjusting circuit

receives the cathode potentials in turn *via* switch section S_K. Any differences, e, are amplified and applied *via* S_G to the appropriate grids so as to modify the cathode currents and reduce all e's towards zero.

Each triode possesses a memory capacitor which holds the grid potential steady between adjustments. Every adjustment modifies the potentials over the whole network, and the final equilibrium is thus approached in stages over several cycles.

The amplifier is conveniently a.c.-coupled through input and output capacitors C_K and C_G respectively, which are large enough to maintain essentially steady potentials during equilibrium operation. Once every cycle, or more often if desired, a zeroing operation is performed which sets the potentials on these capacitors and allows them to be considered as batteries for purposes of analysis. In Fig. 3, zeroing occurs as the switch wipers reach position O, and gives

$$V_K = V_G = F_0 + V_b - V_a \tag{7}$$

where F_0 is the output of the function generator when its input is shorted (Fig. 5) and V_a is a potential which can be adjusted manually to alter the zero of the function.

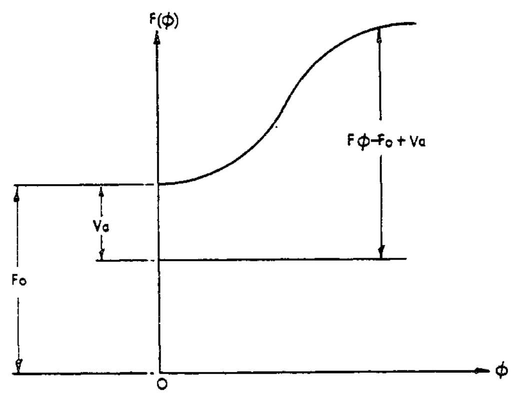

Fig. 5. Typical function

When equilibrium has been established, all the triodes carry currents $i_n = V_n/R$ such that

$$F(\phi_n) - V_n + V_b - V_K = e \qquad (8)$$

for every triode element, and, if the amplifier gain is high, the errors e approximate to zero. Equations (7) and (8) give the equilibrium cathode potentials

$$V_n = F(\phi_n) - F_0 + V_a \text{ approx.} \qquad (9)$$

Thus the common h.t. potential V_b which appears in every inspection of cathode potential, is opposed by an exactly equal potential on C_K and does not affect the operation. Similarly, steady potentials in the function generator output and the amplifier input or output have no effect. This use of a.c. coupling allows a very simple amplifier to be used, eliminates amplifier drift effects and permits a.c. amplification of the function within the function generator. The input of the function generator must, of course, be directly coupled.

SOME PRACTICAL DETAILS

The foregoing remarks assume that each memory capacitor exactly retains its potential between adjustments. In fact, however, grid current in the triodes and capacitor leakage cause slow drifts which must be corrected by continuing to revolve the switches after balance has been reached.

Most valves are found to have grid current well below $0 \cdot 1\ \mu$A, and $1\ \mu$F memory capacitors give a drift rate less than $0 \cdot 1$ V/s which is low enough for most purposes if the cycle time is less than about one second. Selection of valves or faster cycling would reduce the drift rate or alternatively permit a considerable reduction in the value of these capacitors. The effects of leakage resistance become negligible if the potential drift over a cycle from this cause is below say $0 \cdot 1\%$ of the actual potential. This requirement is met if the self time constant of each capacitor exceeds say 1000 times the cycle duration, and good quality paper capacitors used at room temperature are quite adequate for durations up to a few seconds.

The above slow drifts are corrected by small signals applied automatically during every adjustment. Errors are avoided if C_K and C_G are large enough to pass the necessary currents while still maintaining substantially constant potentials. Thus C_K may have a time constant with the amplifier input impedance which is of the same order as the time between zeroing operations, while C_G may have the same value as the memory capacitors. If a rapid approach to balance is required, C_G should be larger, e.g. about equal to the total of memory capacitors between zeroing positions. The charge passed through C_G while several valves are adjusted has to be made up during zeroing and, if this is infrequent, the amplifier should have a proportionately low output impedance. Zeroing may be carried out any number of times in a cycle, greater frequency permitting smaller coupling capacitors but requiring more switch positions. Both coupling capacitors must have high leakage resistance since each sustains the h.t. potential, and C_K at least should be a polystyrene or equivalent type.

Undesired transient signals can be prevented from reaching the amplifier or the memory capacitors by performing the switching in the following sequence:

S_ϕ makes
S_K makes
S_G makes
S_G breaks
S_K breaks
S_ϕ breaks

Any type of function generator could be used which has sufficient accuracy and a response which ensures that the function voltage is fully generated during the time between S_ϕ making and S_K making. There are several types which fulfil this not very onerous requirement, and the biased-diode generator which approximates the function by a number of straight-line segments appears very suitable. The diode network may be followed by an a.c.-coupled amplifier if it is necessary to amplify the function voltage, and in any case should have a low output impedance so as to be unaffected by stray currents running to earth from the main amplifier. Its input impedance must be high enough to leave the network potentials substantially unaffected.

The high impedance input connexion to the main amplifier needs to be well screened to avoid pick-up during switching. This can be achieved by building C_K into the amplifier and using a co-axial lead to the switch that has its sheath connected to the low impedance input. The latter may be connected to the amplifier chassis, this being allowed to float. Similar remarks apply to the high impedance input of the function generator but in this case the chassis is earthed.

EXPERIMENTAL MODEL—RESULTS

A model has been made designated "NOLA" (Non-Linear Analyser) which has eight triode elements, each of which is a 12AX7, 12AT7 or 12BH7 double triode. Cathode resistors are inter-changeable and of the order 10 kΩ; the memory capacitors are each $1\ \mu$F, but smaller values would probably suffice.

Switching is effected by three 44-position switch sections, ganged together and turned by hand. A fourth section is included for other purposes, and the correct switching sequence is obtained by using studs joined together in twos, threes and fours while omitting the remaining studs in every five.

The amplifier has a single stage giving a gain which is variable up to about 50, and a cathode follower output. A five-section biased diode square root function generator has been used for some tests, and this also has a cathode follower output. Other tests have been made with no function generator, i.e. a function of $\times 1$.

Solutions are generally reached, to the extent that no further change can be observed, within about 10 turns of the switch. They are held by continuing to revolve the switch and drift slowly if it is not turned.

As a test problem, equation (2) was solved for the case

$$F(\phi) = (2\phi)^{1/2} \quad \text{i.e. } f(\phi) = \frac{1}{h^2}\frac{r}{R}(2\phi)^{1/2} \qquad (10)$$

where $r = 5$ kΩ, $R = 10$ kΩ, $h = X/9$, $X =$ the range of x considered.

If we assume that $\phi = d\phi/dx = 0$ when $x = 0$, the algebraic solution is

$$\phi = 22 \cdot 78\ (x/X)^4 \qquad (11)$$

which is compared with the analogue solution in the following table.

Comparison of analogue solution with equation (11)

x/X	ϕ *by equation* (11)	ϕ *by analogue*
4/9	0·889	0·89 (set manually)
5/9	2·170	2·17
6/9	4·500	4·49
7/9	8·337	8·2
8/9	14·22	14·2
9/9	22·78	22·6
10/9	34·72	34·6
11/9	50·83	50·7
12/9	72·00	71·9
13/9	99·17	99·2 (set manually)

All resistors were to $\pm 0\cdot 1\%$ tolerance, but the voltage measurements were made on an ordinary high-resistance voltmeter. Fig. 6 shows some further analogue solutions where the boundary values of ϕ have been chosen arbitrarily and $\phi \neq 0$ when $d\phi/dx = 0$. This case has so far resisted exact analysis, but an approximate solution, obtained for x as a power series in ϕ shows the upper curve to be accurate to about 1% at its worst point.

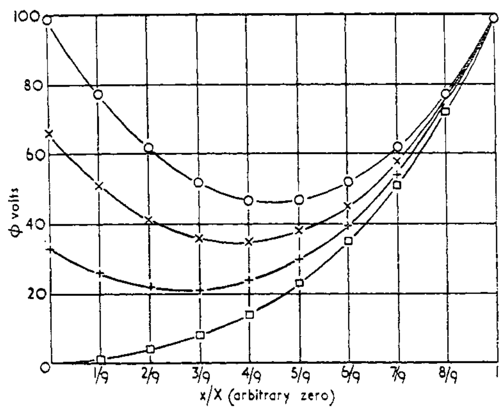

Fig. 6. Analogue solutions of equation (10)

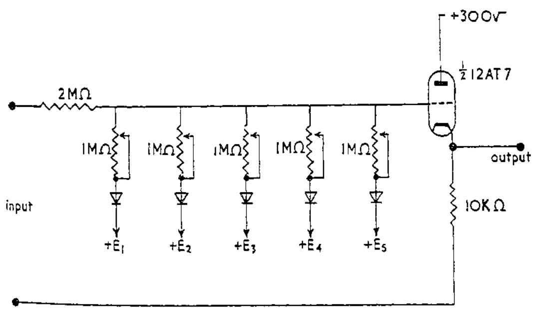

Fig. 7. Square-root function generator

Fig. 7 shows the circuit of the function generator which comprised five biased diodes and a cathode follower output. The break points were set by adjusting E_{1-5}, and the slopes of the intermediate segments by adjusting the 1 MΩ potentiometers. Voltages were measured on the same voltmeter, outputs being taken from a datum given by shorting the input. Values of inputs and outputs at the break points were:

Input (volts)	0	3	8	18	32	50	72
Output above datum (volts)	0	2·45	4	6	8	10	12

POTENTIALITIES OF THE METHOD

The currents withdrawn from the network can be made to depend on the local values of x and y by pre-adjusting the cathode resistors, enabling equations of the form

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = f(x\,y)\,.\,f(\phi) \qquad (12)$$

to be solved. Circuit modifications should enable the potential differences between nodes to be used to control the currents, permitting $\partial\phi/\partial x$ and $\partial\phi/\partial y$ to be introduced on the right-hand side of the above equation.

The addition of another switch section enables the triode elements to be separated. If the current through each is made a function of the potential across it, the elements become a set of non-linear resistors which may be interconnected to solve, for example, flow problems in networks of pipes.

Similarly it is possible to set up currents which are related to potentials existing within a rotating machine by means of a commutator instead of many slip rings.

The accuracy with which Poisson type equations are solved should approach that achieved by Liebmann if a suitably accurate function generator is used. Extension to two dimensions requires the development of a three-pole switch having 100 or more ways, the correct switching sequence being given inherently by the design of the contacts.

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